Equivalence of new-direct and new-indirect Monte Carlo methods

L. Campioni and P. Vestrucci

University of Bologna - Italy

Introduction

- **RAMS**

  Unreliability estimate of highly reliable systems

- **MONTE CARLO METHOD**

- *Traditional* Direct and Indirect MC
- *New* direct and Indirect MC(*)
- System failure function appr.(*)

VARIANCE

REDUCTION

TECHNIQUE

(*) Developed by the authors

Nomenclature

$p_k(t)$	$\tilde{p}_k(t)$	Unbiased and biased *failure* pdf relevant to the *k* component
$p_{sys}(t)$	$\tilde{p}_{sys}(t)$	Unbiased and biased system *transition* pdf
	$g_{sys}(t)$	System failure pdf
$f_{sys}(t)$	$\tilde{f}_{sys}(t)$	Unbiased and biased system *failure* function
$Q_k(t)$	$\tilde{Q}_k(t)$	Unbiased and biased *unreliability* of *k* component
$S_k(t)$	$\tilde{S}_k(t)$	Unbiased and biased system *reliability*

Hypotheses

- **Components**
  - **are independents each other**
  - **have only two states**

- **Variance Reduction Technique**
  - **Importance Sampling**
  - **Each component is associated with only one biasing parameter**

$p_k(t)$

Traditional approaches limits

TRADITIONAL APPROACHES:

based upon the knowledge of

- $p_k(t)$ and
- p_{sys} (the system transition pdf, for the indirect one).

REMARKS:

- they do not use the system failure pdf, g_{sys}(t), where:
- $g_{sys}(t)dt$ is the probability that the system fails between t and $t+dt$.
- this implies that the weighting procedure is constructed "inductively", without a robust general frame

System failure pdf: a heuristic approach (1/4)

paradoxical definition of component:

- the component is the elementary part of the system which is responsible at least in one case of the system failure, in the sense that the system fails between t and *t+dt*, when the component fails at that time, provided (in general) that other components failed before *t*.

- each component *k* can be responsible of the system failure: at least a cut set is accomplished as a consequence of the failure of *k*.

System failure pdf: a heuristic approach (2/4)

The probability that the system fails between t and $t+dt$ due to the failure of the component k, provided that:

- a cut set including the components $q,r,s,..$is completed and
- components u,v,w *are not failed*,

is given by:

$$p_k(t)\times(Q_q(t)Q_r(t)Q_s(t)...)\times(S_u(t)S_v(t)S_w(t)...)dt$$

.

REMARK: this is the probability of a family of sequences:

q, r, s,… failed any time before t

System failure pdf: a heuristic approach (3/4)

Of course, the failure of *k* can be the last transition of several families –say *l*- of system failure sequences, so that the probability density of all of them is

- $$p_k(t)\sum_l \left\{ \left[Q_q(t) Q_r(t) Q_s(t) \ldots \right] \left[S_u(t) S_v(t) S_w(t) \ldots \right] \right\}_l dt$$

This is the probability that the system fails between *t* and *t+dt* due to the failure of *k* at that time, taking into account all the possible cut sets *l*

System failure pdf: a heuristic approach (4/4)

Finally, the system failure pdf is given by the sum of the previous quantity relevant to all the components:

•

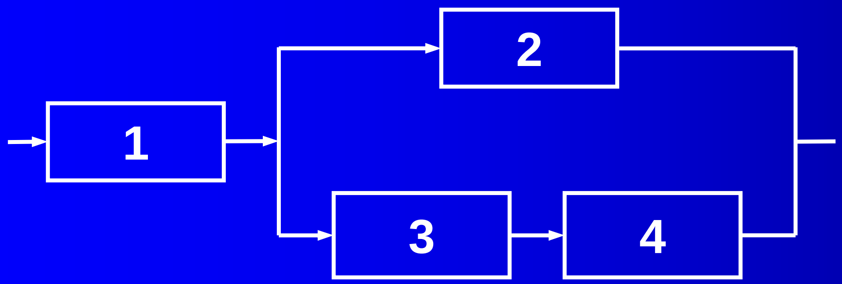
$$g_{sys}(t)=\sum_{k} p_k(t)\sum_{l}\left\{\left[Q_q(t)Q_r(t)Q_s(t)\ldots\right]\left[S_u(t)S_v(t)S_w(t)\ldots\right]\right\}_l$$

REMARKS

- an explicit definition of $g_{sys}(t)$ requires the identification of the l cut sets (not just minimal cut sets) relevant to component k,

- the implicit form given above is sufficient for Monte Carlo simulation.

Case Study



Reliability Block Diagram

Natural failure distributions: exponential

$$p_k\left(t_k\right) = \lambda_k \exp\left[-\lambda_k \; t_k\right]$$

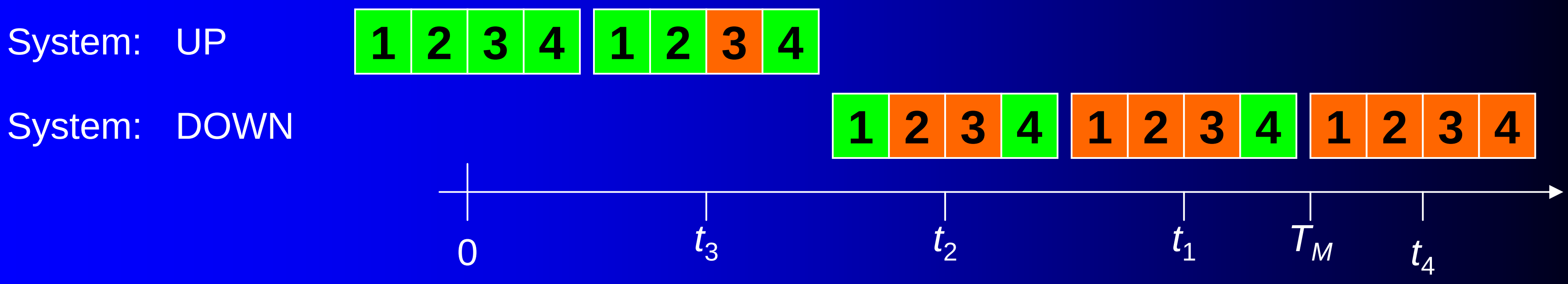
Failure rate [h^{-1}]

Biasing failure distributions: exponential

$$\tilde{p}_k\left(t_k, \lambda_k^*\right) = \lambda_k^* \exp\left[-\lambda_k^* \; t_k\right]$$

Biasing parameter [h^{-1}]

Example of failure sequence



Where:

-T_M is the mission time

- t_1 , t_2 , t_3 , t_4 are the failure times of the components

*

Traditional direct approach

- transition times sampled from the pdf's

$$(i_1, t_1), (i_2, t_2), \ldots (i_{N_C}, t_{N_C})$$

where

$t_1 < t_2 < \ldots. < t_k < T_M < t_{k+1} < \ldots < t_{Nc}$ is the failure times sequence

$i_1, i_2, \ldots i_{Nc}$ are the 1th, the 2nd … component failing

- the resulting weight is

$$w = \prod_{m=1}^{k} \frac{p_{i_m}(t_m)}{\tilde{p}_{i_m}(t_m)} \prod_{n=k+1}^{N_C} \frac{S_{i_n}(T_M)}{\tilde{S}_{i_n}(T_M)}$$

- in our example

$$w = \frac{p_3(t_3)}{\tilde{p}_3(t_3)} \frac{p_2(t_2)}{\tilde{p}_2(t_2)} \frac{S_1(T_M)}{\tilde{S}_1(T_M)} \frac{S_4(T_M)}{\tilde{S}_4(T_M)}$$

New direct approach

- The new direct Monte Carlo follows in a very straightforward way from the failure system pdf:

$$g_{sys}(t)=\sum_{k} p_k(t)\sum_{l}\left\{\left[Q_q(t)Q_r(t)Q_s(t)\ldots\right]\left[S_u(t)S_v(t)S_w(t)\ldots\right]\right\}_l$$

- once the sampling and the ordering steps have been done, it is apparent that a family sequence is selected
- the history weight resulting from this approach is

$$w=\frac{p_k(t_k)Q_q(t_k)Q_r(t_k)Q_s(t_k)\ldots S_u(t_k)S_v(t_k)S_w(t_k)\ldots}{\tilde{p}_k(t_k)\tilde{Q}_q(t_k)\tilde{Q}_r(t_k)\tilde{Q}_s(t_k)\ldots \tilde{S}_u(t_k)\tilde{S}_v(t_k)\tilde{S}_w(t_k)\ldots}$$

- for the example

$$w=\frac{p_2(t_2)Q_3(t_2)S_1(t_2)S_4(t_4)}{\tilde{p}_2(t_2)\tilde{Q}_3(t_2)\tilde{S}_1(t_2)\tilde{S}_4(t_2)\ldots}$$

Traditional indirect approach

The random walk is carried out by sampling the time at which the system undergoes the first transition; then it is sampled the component which fails, and so on, up to the *k*-th transition occurring before *TM*, leading to the system failure

For our example:

$$w = \left[\frac{p_{sys}(t_3)}{\tilde{p}_{sys}(t_3)} \frac{q(3 \mid t_3)}{\tilde{q}(3 \mid t_3)} \right] \cdot \left[\frac{p_{sys}(t_2 \mid (3, t_3))}{\tilde{p}_{sys}(t_2 \mid (3, t_3))} \frac{q(2 \mid (3, t_3), t_2)}{\tilde{q}(2 \mid (3, t_3), t_2)} \right]$$

$q(3|t_3)$ is the probability that the transition occurring at $t3$ is that of component 3 an so on

New indirect approach (1/2)

- inductive procedure: at $t= t_1$, system and components are on:

$$\Pi_k S_k(t_1), \qquad k=1,\ldots,NC.$$

- and we have the component q failure, given that it was functioning in $(0,t_1)$ => the q failure rate : $p_q(t_1)/S_q(t_1)$ =>

$$\Pi_k S_k(t_1) \cdot \; p_q(t_1)/S_q(t_1) = p_q(t_1) \Pi_{k \neq q} S_k(t_1)$$

is the probability that the system has the first transition at $t=t_1$ due to the failure of component q.

New indirect approach (2/2)

- the probability that the system undergoes 2 transitions due to the components *q* (which fails in $0<t_1<t_2$) and *r* (which fails al t_2), is

$$\prod_{\substack{k=1 \\ k\neq q}}^{NC} S_k(t_2) Q_q(t_2) \times p_r(t_2) / S_r(t_2) = p_r(t_2) Q_q(t_2) \prod_{\substack{k=1 \\ k\neq q \\ k\neq r}}^{NC} S_k(t_2)$$

- following this procedure it is to get the previous direct formulation for the probabilities and, consequently, for the weights

System Failure Function app.

System failure function: gives the state of the system as a function of all component failure functions

$$f_{sys}\left(f_1, f_2, \ldots f_{N_C}\right) = \begin{cases} 1, & down \\ 0, & up \end{cases}$$

For our example,

$$f_{sys} = f_1 + f_2 \cdot f_3 + f_2 \cdot f_4 - f_1 \cdot f_2 \cdot f_3 - f_1 \cdot f_2 \cdot f_4 - f_2 \cdot f_3 \cdot f_4 + f_1 \cdot f_2 \cdot f_3 \cdot f_4$$

and the weight of the history is given by combining according to the system failure function only the weights of the failed components.

For our example,

$$\tilde{f}_{sys} = \frac{p_1(t_1)}{\tilde{p}_1(t_1)} + \frac{p_2(t_2)}{\tilde{p}_2(t_2)} \cdot \frac{p_3(t_3)}{\tilde{p}_3(t_3)} - \frac{p_1(t_1)}{\tilde{p}_1(t_1)} \cdot \frac{p_2(t_2)}{\tilde{p}_2(t_2)} \cdot \frac{p_3(t_3)}{\tilde{p}_3(t_3)}$$

Unbiased MC

	#1	#2	#3	#4
$\lambda_k[h^{-1}]$	1.0E-5	5.0E-5	1.0E-4	5.0E-3
Q_k	9.9995E-5	4.99875E-4	9.995E-4	4.87706E-2
$\lambda^*_k[h^{-1}]$	0.159	0.159	0.159	0.161

Mission time

$$T_M = 10\,h$$

System unreliability : exact

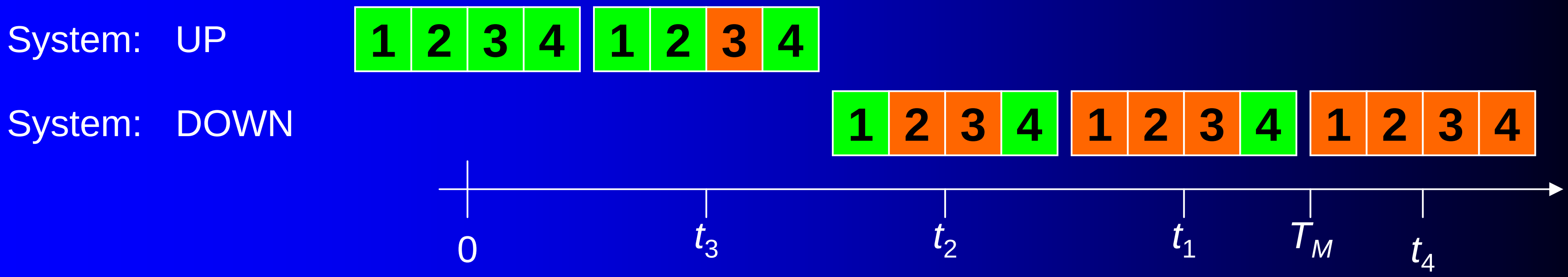
$$Q_{\text{sys}} = 1.24847\ \mathrm{E}-4$$

System unreliability : MC estimate

$$Q_{\text{sys,MC}} = 1.30\ \mathrm{E}-4 \qquad N_H = 100\,000$$

$$\varepsilon_{\text{abs}} = 5.15\,\mathrm{E}-6 \qquad \varepsilon_{\text{rel}} = 4.13\,\mathrm{E}-2$$

Example of failure sequence



Direct approach

$$w = \frac{p_3(t_3)}{\tilde{p}_3(t_3)} \frac{p_2(t_2)}{\tilde{p}_2(t_2)} \frac{S_1(T_M)}{\tilde{S}_1(T_M)} \frac{S_4(T_M)}{\tilde{S}_4(T_M)}$$

Indirect approach

$$w = \left[\frac{p_{sys}(t_3)}{\tilde{p}_{sys}(t_3)} \frac{q(3 \mid t_3)}{\tilde{q}(3 \mid t_3)} \right] \cdot \left[\frac{p_{sys}(t_2 \mid (3,t_3))}{\tilde{p}_{sys}(t_2 \mid (3,t_3))} \frac{q(2 \mid (3,t_3),t_2)}{\tilde{q}(2 \mid (3,t_3),t_2)} \right]$$

New dir/indirect approaches *

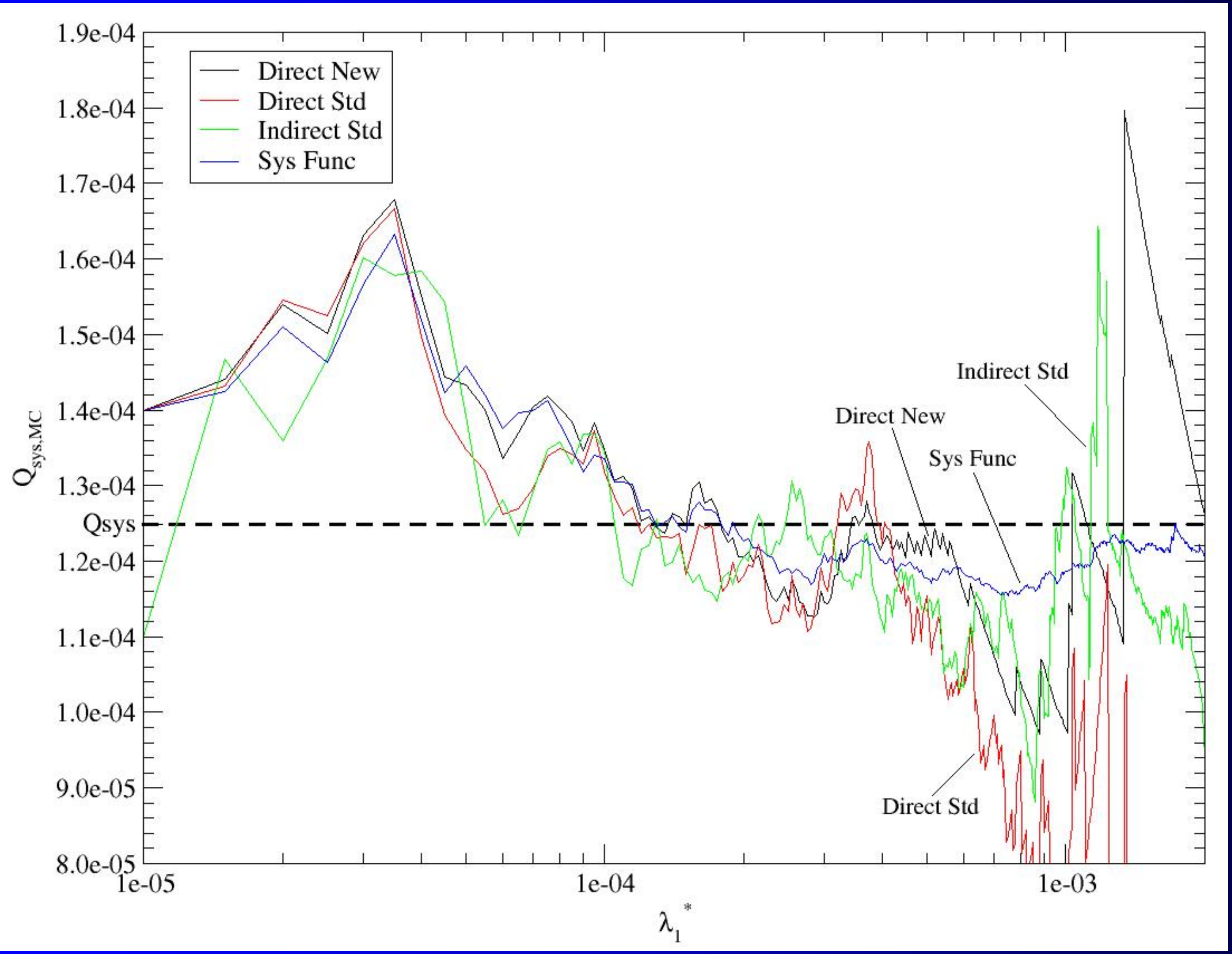
$$w = \frac{p_2(t_2)}{\tilde{p}_2(t_2)} \frac{Q_3(t_2)}{\tilde{Q}_3(t_2)} \frac{S_1(t_2)}{\tilde{S}_1(t_2)} \frac{S_4(t_2)}{\tilde{S}_4(t_2)}$$

System failure function app*

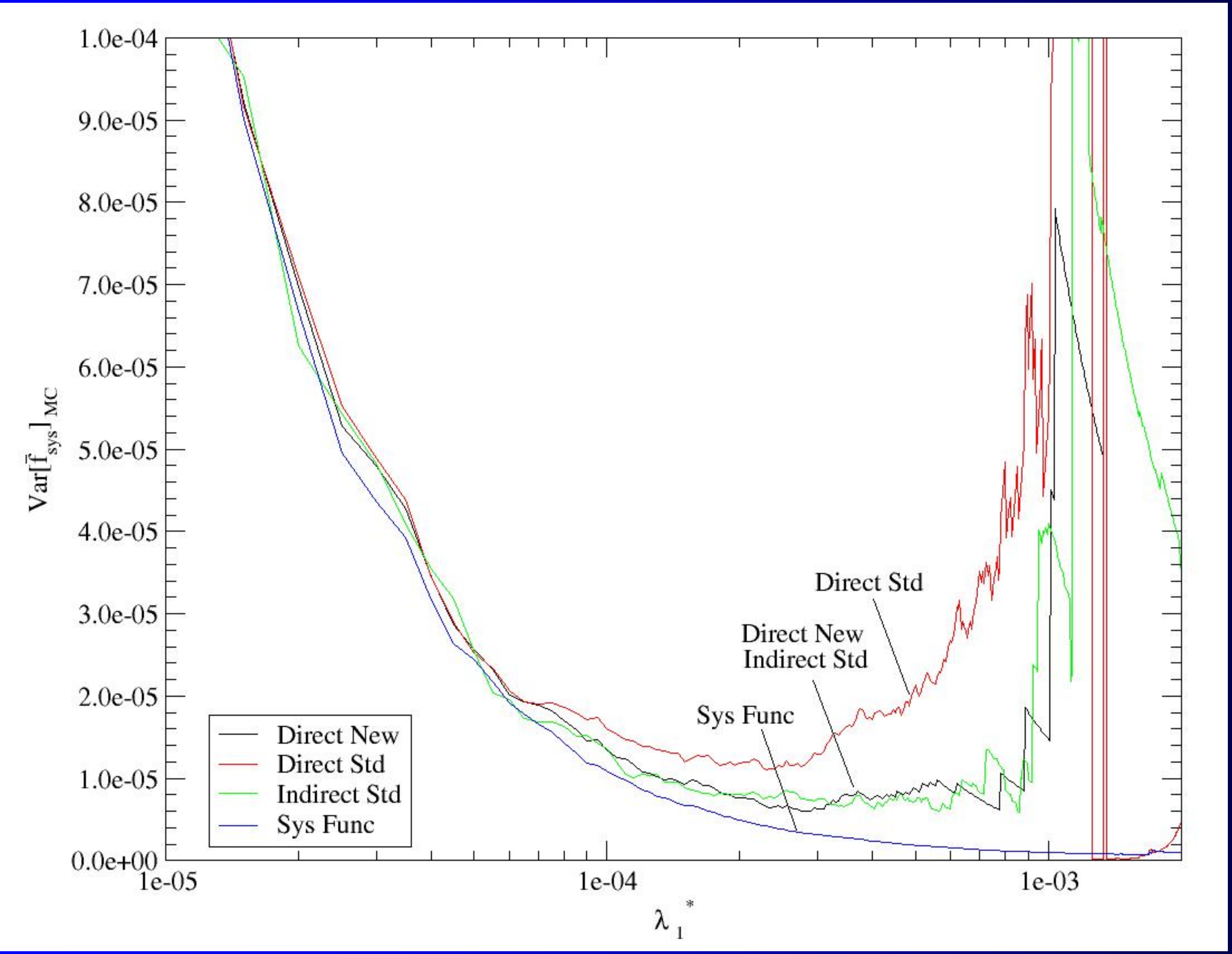
$$w = \frac{p_1(t_1)}{\tilde{p}_1(t_1)} + \frac{p_2(t_2)}{\tilde{p}_2(t_2)} \cdot \frac{p_3(t_3)}{\tilde{p}_3(t_3)} - \frac{p_1(t_1)}{\tilde{p}_1(t_1)} \cdot \frac{p_2(t_2)}{\tilde{p}_2(t_2)} \cdot \frac{p_3(t_3)}{\tilde{p}_3(t_3)}$$

*** Developed by the authors**

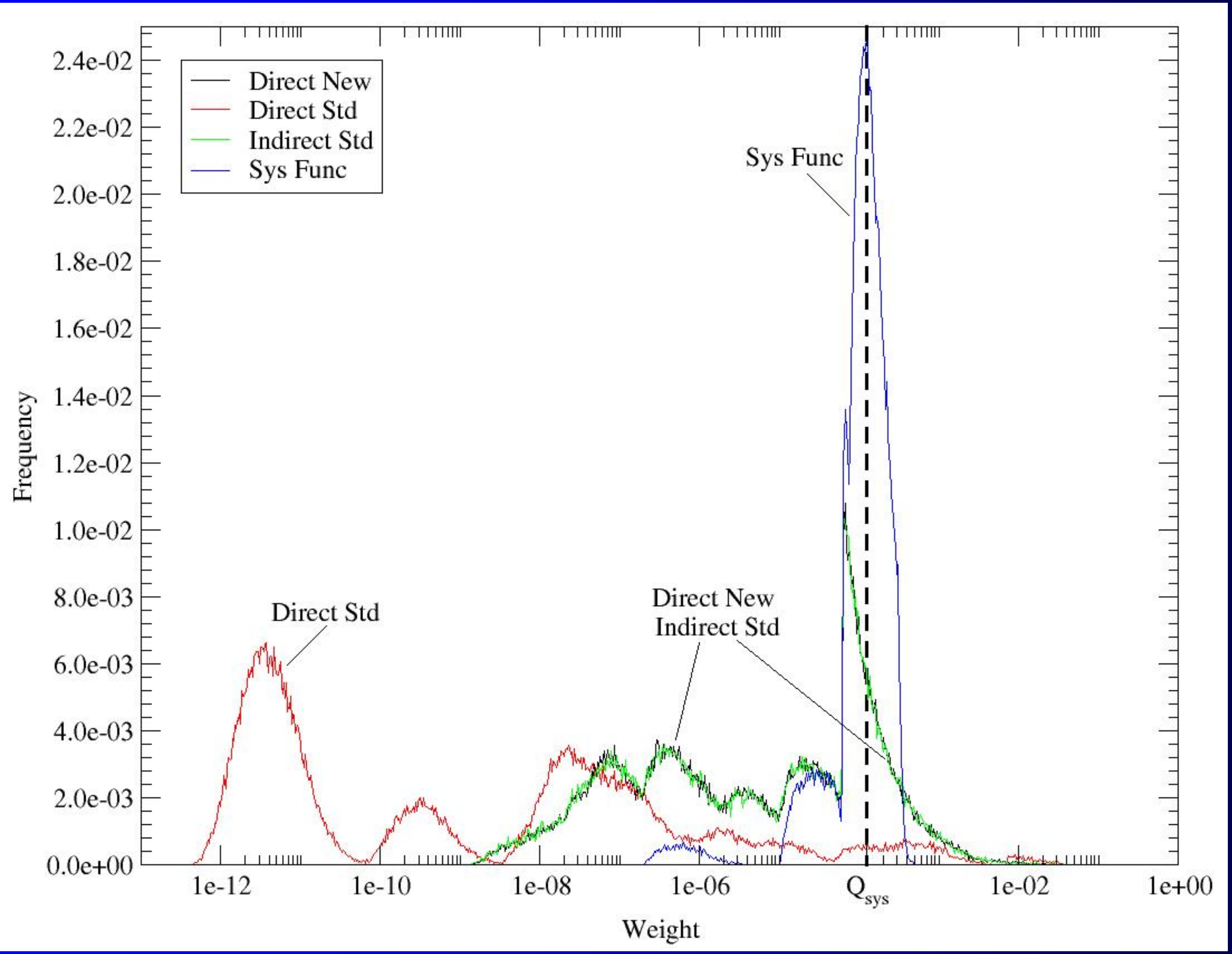
Comparison: unreliability estimates



Comparison: variance estimates



Comparison: failure weights



Conclusions

1. Direct and indirect methods: these must be equivalent, as far as they use the same approach based on the component and system transition probabilitiy density functions.

   New direct/indirect and trad. indirect unreliability, variance and weights are almost overlapped, while with the standard direct we get evidence of significant differences.

2. New methods are more efficient as far as a family of histories is simulated each time

3. The variance reduction techniques for MC system analysis seems to be a field in which a deep investigation is still necessary